

ALMOST SURE STABILITY OF PARTIAL SUMS OF UNIFORMLY BOUNDED RANDOM VARIABLES

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ABSTRACT. Suppose a_1, a_2, \dots is a sequence of real numbers with $a_n \rightarrow \infty$. If $\limsup (X_1 + \dots + X_n)/a_n = \alpha$ a.s. for every sequence of independent nonnegative uniformly bounded random variables X_1, X_2, \dots satisfying some hypothesis condition A, then for every (arbitrarily-dependent) sequence of nonnegative uniformly bounded random variables Y_1, Y_2, \dots , $\limsup (Y_1 + \dots + Y_n)/a_n = \alpha$ a.s. on the set where the conditional distributions (given the past) satisfy precisely the same condition A. If, in addition, $\sum a_n^{-2} < \infty$, then the assumption of nonnegativity may be dropped.

1. Introduction. The purpose of this note is to provide partial answers to the question of when the following statement (S1) is true in the case where “property B” is a property reflecting the limiting behavior of the stabilized (or normalized) partial sums.

(S1) If every sequence of *independent* random variables having property A has property B almost surely, then *every* sequence of random variables has property B almost surely on the set where the conditional distributions (given the past) have property A.

Of course (S1) is not true for all choices of properties A and B, but it is sometimes true in rather general contexts. In [4], for example, it was shown that (S1) is true whenever “property B” is “the partial sums converge”, *regardless* of property A. In the present note, a partial analysis of (S1) is given for conclusions concerning the limiting behavior, not of the partial sums S_n , but of the stabilized partial sums S_n/a_n for suitable normalizing constants $\{a_n\}$.

As in [4], $\mathcal{Y} = (Y_1, Y_2, \dots)$ is a sequence of random variables on a probability triple $(\Omega, \mathfrak{A}, P)$, $S_n = Y_1 + \dots + Y_n$, and \mathcal{F}_n is the sigma field generated by Y_1, \dots, Y_n . Let $\pi_n(\cdot, \cdot)$ be a regular conditional distribution for Y_n given \mathcal{F}_{n-1} , and let $\Pi = (\pi_1, \pi_2, \dots)$. Let \mathfrak{B} denote the Borel σ -field on \mathbf{R} , and \mathfrak{B}^∞ the product σ -field on \mathbf{R}^∞ ; let $\mathcal{P}(\mathbf{R})$ denote the space of probability measures on $(\mathbf{R}, \mathfrak{B})$, and let $\mathcal{C} = \mathcal{P}(\mathbf{R}) \times \mathcal{P}(\mathbf{R}) \times \dots$. (As mentioned in [4], it might help the reader to think of \mathcal{Y} as a random

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element of \mathbf{R}^∞ , and of Π as a random element of \mathcal{C} .) $\mathcal{L}(X)$ is the distribution of the random variable X .

Let $B \in \mathfrak{B}^\infty$. In this notation, the formalization of statement (S1) is the following statement (S2) of [4].

- (S2) If $A \subset \mathcal{C}$ is such that $(X_1, X_2, \dots) \in B$ a.s. whenever X_1, X_2, \dots are independent and $(\mathcal{L}(X_1), \mathcal{L}(X_2), \dots) \in A$, then for arbitrary $\mathfrak{U}, \mathfrak{V} \in B$ a.s. on the set where $\Pi \in A$.

DEFINITION 1. $\mathcal{K} = \{B \in \mathfrak{B}^\infty: (S2) \text{ is true for all } A \subset \mathcal{C}\}$.

Throughout this note, a_1, a_2, \dots and b_1, b_2, \dots are sequences of real numbers with $a_n \rightarrow \infty$ and $\sum^\infty b_n^{-2} < \infty$, and $\vec{r} = (r_1, r_2, \dots)$. The main positive result in this paper is the following theorem.

THEOREM 1. For all $M > 0$, all $-\infty < a < b < \infty$, and all $D \subset [-\infty, \infty]$,

- (i) $B = B_0^M = \{\vec{r} \in \mathbf{R}^\infty: 0 \leq r_i \leq M \text{ for all } i, \text{ and } \limsup_{n \rightarrow \infty} (r_1 + \dots + r_n)/a_n \in D\} \in \mathcal{K}$;

and

- (ii) $B = B_a^b = \{\vec{r} \in \mathbf{R}^\infty: a \leq r_i \leq b \text{ for all } i, \text{ and } \limsup_{n \rightarrow \infty} (r_1 + \dots + r_n)/b_n \in D\} \in \mathcal{K}$.

Moreover, the analogs of (i) and (ii) with \limsup replaced by \liminf or \lim also hold.

In contrast to the conclusions of Theorem 1, examples will be given in §4 to show that

$$(1) \quad B = \left\{ \vec{r} \in \mathbf{R}^\infty: -1 \leq r_i \leq 1 \text{ for all } i, \text{ and } \limsup_{n \rightarrow \infty} (r_1 + \dots + r_n) / (n \log \log(n/2))^{1/2} = 1 \right\} \notin \mathcal{K};$$

and that

$$(2) \quad B = \left\{ \vec{r} \in \mathbf{R}^\infty: \lim_{n \rightarrow \infty} (r_1 + \dots + r_n)/n \neq 0 \right\} \notin \mathcal{K}.$$

2. Applications of Theorem 1. Theorem 1(i) essentially says that any theorem for nonnegative uniformly bounded independent random variables which concludes “ $\limsup_{n \rightarrow \infty} S_n/a_n \in D$ a.s.” may immediately be generalized into a conditional version of that result. For example, suppose one just learned the following fact.

- (3) If X_1, X_2, \dots are independent random variables taking values in $[0, 1]$, and if $P(X_n \leq x) \geq 1 - x^{n-1}$ for all $x \in [0, 1]$ and all n , then

$$\limsup (X_1 + \dots + X_n) / \log n \leq 1 \quad \text{a.s.}$$

Applying Theorem 1(i) with $a_n = \log n$, $M = 1$, and $D = [0, 1]$ allows the generalization of (3) given by (4).

- (4) If Y_1, Y_2, \dots are (arbitrarily-dependent) random variables taking values in $[0, 1]$, then $\limsup_{n \rightarrow \infty} S_n / \log n \leq 1$ a.s. on the set where $P(Y_n \leq x | \mathcal{F}_{n-1}) \geq 1 - x^{n-1}$ for all $x \in [0, 1]$ and all n .

Clearly it is not difficult to prove (4) using the conditional versions of the arguments used to establish (3), as is often the case; the point of Theorem 1 is that this is not necessary.

As an application of Theorem 1(ii), one may prove Levy's martingale strong law of large numbers [5, §69, p. 250] by proving the special case of independence, and then applying Theorem 1(ii) with $b_n = n$ and $D = \{0\}$. For a stronger similar result, consider first the following theorem (5).

- (5) [6, Theorem 2.8.1]. Let X_1, X_2, \dots be a sequence of independent, mean zero uniformly bounded random variables. Then $(X_1 + \dots + X_n)/n^{1/2+\varepsilon} \rightarrow 0$ a.s. for each $\varepsilon > 0$.

Applying Theorem 1(ii) one immediately has the following martingale generalization of (5).

- (6) Let S_1, S_2, \dots be a martingale with uniformly bounded increments. Then $S_n/n^{1/2+\varepsilon} \rightarrow 0$ a.s. for each $\varepsilon > 0$.

(Stronger results than (6) are known; see, for example, Chow's result [2 or 7, Theorem 3.3.1].) It should be noted that the classical formulations of most laws of the iterated logarithm do not fit the framework of Theorem 1; they usually involve some form of centering which violates the nonnegativity assumption of (i), and clearly $b_n = (n \log \log n)^{1/2}$ does not satisfy the $\Sigma^\infty b_n^{-2} < \infty$ hypothesis of (ii).

Although the above applications all involve conditional *moment* hypotheses, the "hypothesis" set A in (S2) need neither be measurable nor involve (conditional) moments. (For a similar nonmoment application, the reader may see [4, Theorem 2].) It should also be remarked that technically speaking the set D in Theorem 1 may also be nonmeasurable, although in most applications, D will simply be a point or interval.

3. Proof of Theorem 1. The argument below closely parallels that in the proof of [4, Theorem 1]; the crucial difference being the use of Lemma 2 in place of the three-series and Borel-Cantelli arguments of [4]. Without loss of generality, assume $(\Omega, \mathfrak{A}, P)$ is complete.

LEMMA 1 [4, LEMMA 1]. (S2) \Leftrightarrow (S3).

- (S3) $P(\{\omega: P_{\Pi(\omega)}(B) = 1\} \cap \mathfrak{U} \notin B) = 0$ for all \mathfrak{U} , where $P_{\Pi(\omega)}$ is the product measure $\pi_1(\omega) \times \pi_2(\omega) \times \dots$ on $(\mathbf{R}^\infty, \mathfrak{B}^\infty)$.

REMARKS. In Lemma 1, the "hypothesis" set A of (S2) has in effect been replaced by the set representing *all* possible sufficient conditions for the conclusion " $(X_1, X_2, \dots) \in B$ " to hold; one direction of the equivalence uses the completeness of the measure space to guarantee all subsets of null sets are measurable. In terms of the class \mathfrak{K} , Lemma 1 says

- (7) $\mathfrak{K} = \{B \in \mathfrak{B}^\infty: P(\{\omega: P_{\Pi(\omega)}(B) = 1\} \cap \mathfrak{U} \notin B) = 0 \text{ for all } \mathfrak{U}\}.$

LEMMA 2. Let Y_1, Y_2, \dots be a sequence of uniformly bounded (arbitrarily-dependent) random variables, and let $s_n = \Sigma_1^n E(Y_j | \mathfrak{F}_{j-1})$. Then

- (i) $\limsup_{n \rightarrow \infty} S_n/b_n = \limsup_{n \rightarrow \infty} s_n/b_n$ a.s.; and
(ii) if $Y_i \geq 0$ for all i , then $\limsup_{n \rightarrow \infty} S_n/a_n = \limsup_{n \rightarrow \infty} s_n/a_n$ a.s.

PROOF. (i) follows from a martingale result of Chow [2 or 7, Theorem 3.3.1].

(ii) follows easily (since $a_n \rightarrow \infty$) from a generalization of a result of Dubins and Freedman [3] which can be found in Brown [1], namely:

(8) If Y_1, Y_2, \dots are nonnegative uniformly bounded random variables, then $\lim_{n \rightarrow \infty} S_n/s_n < \infty$ a.s., and $= 1$ almost surely on the set $s_n \rightarrow \infty$. \square

The following example shows that the “uniform boundedness” assumption in Lemma 2 may not be replaced by “tightness”.

EXAMPLE 1. Let Y_1, Y_2, \dots be independent with

$$P(Y_n = n^2) = n^{-2} = 1 - P(Y_n = 0).$$

Then $s_n/n \equiv 1$, but $\limsup_{n \rightarrow \infty} S_n/n = 0$ a.s. by the Borel-Cantelli Lemma.

PROOF OF THEOREM 1. The argument for (i) only will be given; that for (ii) and \liminf or \lim are similar.

Fix $M > 0$ and $D \subset [-\infty, \infty]$. By (7), it suffices to show that B satisfies (S3), where $B = B_0^M = \{\vec{r} \in \mathbf{R}^\infty: 0 \leq r_i \leq M \text{ for all } i, \text{ and } \limsup_{n \rightarrow \infty} (r_1 + \dots + r_n)/a_n \in D\}$. Fix $\mathfrak{U} = (Y_1, Y_2, \dots)$.

Let $(\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P})$ be a copy of $(\Omega, \mathfrak{A}, P)$, and (enlarging this new space if necessary) for each $\omega \in \Omega$, let $Z_1(\omega), Z_2(\omega), \dots$ be a sequence of independent random variables on $(\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P})$ with $\mathbb{P}(Z_n(\omega)) = \pi_n(\omega)$. Let $E = \{\omega \in \Omega: \text{support of } \pi_n(\omega) \subset [0, M] \text{ for all } n\}$, and observe that, without loss of generality, it may be assumed that $E = \{\omega \in \Omega: 0 \leq Z_n(\omega) \leq M \text{ everywhere (in } \hat{\Omega}) \text{ for all } n \geq 1\}$. Next, calculate

$$\begin{aligned} & P(\{\omega: P_{\Pi(\omega)}(B) = 1\} \cap \mathfrak{U} \notin B) \\ &= P(E \cap \{\omega: P_{\Pi(\omega)}(B) = 1\} \cap \mathfrak{U} \notin B) \\ &= P\left(\left\{\omega: 0 \leq Z_i(\omega) \leq M \text{ for all } i \text{ and} \right. \right. \\ &\quad \left. \left. \limsup_{n \rightarrow \infty} (Z_1(\omega) + \dots + Z_n(\omega))/a_n \in D \text{ a.s. (in } (\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P}))\right\} \cap \mathfrak{U} \notin B\right) \\ &= P\left(\left\{\omega: 0 \leq Z_i(\omega) \leq M \text{ for all } i \text{ and} \right. \right. \\ &\quad \left. \left. \limsup_{n \rightarrow \infty} (\hat{E}Z_1(\omega) + \dots + \hat{E}Z_n(\omega))/a_n \in D\right\} \cap \mathfrak{U} \notin B\right) \\ &= P\left(E \cap \left\{\omega: \limsup_{n \rightarrow \infty} (EY_1 + E(Y_2 | \mathcal{F}_1)(\omega) \right. \right. \\ &\quad \left. \left. + \dots + E(Y_n | \mathcal{F}_{n-1})(\omega))/a_n \in D\right\} \cap \mathfrak{U} \notin B\right) \\ &= 0, \end{aligned}$$

where the first equality in (9) follows by the definitions of E and B ; the second by the definitions of $Z_n(\omega)$ and B ; the third by Lemma 2(ii) (recall that $Z_1(\omega), Z_2(\omega), \dots$ are independent in $(\hat{\Omega}, \hat{\mathfrak{A}}, \hat{P})$ for each ω); the fourth by the definition of $Z_n(\omega)$ and π_n ; and the last by Lemma 2(ii) again and the definitions of B and E . \square

4. Examples establishing (1) and (2).

EXAMPLE 2 [4]. Let $Y_n = S_n - S_{n-1}$, where S_1, S_2, \dots are iid, $P(S_n = 0) = P(S_n = 1) = \frac{1}{2}$. Then for $a_n = (n \log \log(n/2))^{1/2}$ and $D = \{1\}$, $\limsup_{n \rightarrow \infty} S_n/a_n = 0 \notin D$ a.s., but $P_{\Pi(\omega)}(B) = 1$ a.s. for

$$B = \left\{ \vec{r} \in \mathbb{R}^\infty : -1 \leq r_i \leq 1 \text{ for all } i, \text{ and } \limsup_{n \rightarrow \infty} (r_1 + \dots + r_n)/a_n = 1 \right\}.$$

Via (7), this proves (1).

EXAMPLE 3. Let S_1, S_2, \dots be independent with $P(S_n = \sqrt{n}) = P(S_n = -\sqrt{n}) = \frac{1}{2}$ and let $Y_n = S_n - S_{n-1}$ ($Y_1 = S_1$). From the definition of S_n , one has that $\lim S_n/n = 0$ a.s. There are only two possible conditional laws for π_n , namely

$$\pi_n^{(+)} = \left(\delta(\sqrt{n} + \sqrt{n-1}) + \delta(-\sqrt{n} + \sqrt{n-1}) \right) / 2,$$

and

$$\pi_n^{(-)} = \left(\delta(\sqrt{n} - \sqrt{n-1}) + \delta(-\sqrt{n} - \sqrt{n-1}) \right) / 2.$$

Construct $\{Z_n\}$ as in the proof of Theorem 1. Then the *unconditional* distribution of $\{Z_n\}$ is independent with law

$$\begin{aligned} P(Z_n = \sqrt{n} + \sqrt{n-1}) &= P(Z_n = \sqrt{n} - \sqrt{n-1}) = P(Z_n = -\sqrt{n} + \sqrt{n-1}) \\ &= P(Z_n = -\sqrt{n} - \sqrt{n-1}) = \frac{1}{4}. \end{aligned}$$

It will now be shown that

$$(10) \quad (Z_1 + \dots + Z_n)/n \not\rightarrow 0 \quad \text{a.s.}$$

To see (10), consider the following theorem of Revesz [6, p. 65 or 7, p. 167]:

$$(11) \quad \begin{aligned} &\text{If } Z_1, Z_2, \dots \text{ are independent with } |Z_n| \leq n \text{ a.s. and } E(Z_n) = 0 \\ &\text{for all } n \geq 1, \text{ then } (Z_1 + \dots + Z_n)/n \rightarrow 0 \quad \text{a.s. implies} \\ &\lim_{n \rightarrow \infty} [\Sigma^n E(Z_i^2/i^2)]/\log n = 0. \end{aligned}$$

Since $E(Z_n^2) = 2n - 1$ for all n , then $\lim_{n \rightarrow \infty} \Sigma^n E(Z_i^2/i^2)/\log n = 0$ which by (11) implies (10). Since $\lim_{n \rightarrow \infty} S_n/n = 0$ a.s., this establishes (2). (Informally, (2) says that “ S_n/n does not converge to zero” is not a “property B” for which (S1) holds for every property A.)

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